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A New Kumaraswamy Transmuted Exponential Distribution with Applications to Lifetime Data

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SUMMARY

This paper introduces a new generalization of the Kumaraswamy transmuted exponentiated exponential distribution, based on a new family of life time distribution by Mansour et al.(2015) .We refer to the new distribution as Kumaraswamy new transmuted exponential $(Kw-NTE)$ distribution. The new model contains some of lifetime distributions as special cases such as exponentiated exponential, transmuted exponential and exponential distributions. The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. This model is capable of modeling various shapes of aging and failure criteria.

Keywords and phrases: transmutation; survival function; exponentiated exponential; order statistics; maximum likelihood estimation

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1 Introduction

The nature of the methodology utilized as a part of a factual examination depends vigorously on the expected probability model. On account of it, extensive exertion has been exhausted in the improvement of substantial classes of standard probability circulations alongside pertinent measurable philosophies. Be that as it may, there still stay numerous imperative issues where the genuine information does not take after any of the traditional or standard probability models.

For complex electronic and mechanical frameworks, the hazard rate frequently shows non-monotonic (bathtub or upside-down bathtub unimodal) hazard rates (Xie and Lai (2006). models with such hazard rates have pulled in an impressive consideration of scientists in unwavering quality building. In programming unwavering quality, bathtub formed hazard rate is experienced in firmware, and in implanted programming in equipment gadgets. The upside down bathtub shaped failure rate is used in data of motor bus failures (Mudholkar et al. (1995), for optimal burn-in decisions (Block and Savits (2010), Chang (2000), for ageing properties in reliability (Gupta and Gupta (1983), Jiang et al. (2001) and the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

Many of distributions have been made using cumulative distribution function (*cdf*)$G(x)$, probability density function (*pdf*)$g(x)$, or survival function $\bar{G}(x)$ that one can rely on, as a baseline distribution, to introduce new models. The Exponentiated generalization is the first generalization allowing for non-monotone hazard rates, including the bathtub shaped hazard rate. The *cdf* of the new distribution is defined by $F(x) = G^{\alpha}(x)$, where $\alpha > 0$. The exponentiated exponential distribution has been introduced by Ahuja and Nash (1967), and further studied by Gupta and Kundu (1999). The first generalization allowing for nonmonotone hazard rates, including the bathtub shaped hazard rate, is the exponentiated Weibull (*EW*) distribution due to Mudholkar and Srivastava (1993), and Mudholkar et al. (1995). Merovci (2013) introduced transmuted exponentiated exponential distribution. According to the transmutation generalization approach, the *cdf* satisfies the relationship,

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2, \tag{1.1}$$

where $G(x)$ the *cdf* of the baseline distribution.

Mansour et al. (2015a) introduced a modification of the transmutation generalization approach given in 1.1. The proposed modification generalizes the rank of the transmutation map by replacing the constant power by additional parameters. The following definition gives the mechanism of generating a new family of lifetime distributions building on a base model, that is, according to this modification.

Definition 1.1. Let $G(x)$ be the cumulative distribution function (*cdf*) of a non-negative absolutely continuous random variable, $G(x)$ be strictly increasing on its support, and $G(0) = 0$ define a new $cdf, F(x), out of G(x)$ as

$$F(x) = (1 + \lambda)[G(x)]^{\delta} - \lambda[G(x)]^{\alpha}, \tag{1.2}$$

where $\alpha, \delta > 0\, if\, -1 < \lambda < 0\, and\, \alpha > 0, \frac{\alpha}{2} \leqslant \delta \leqslant \frac{5\alpha}{4}\, if\, 0 < \lambda < 1$.

Kumaraswamy (1980) introduced a two-parameter distribution on $(0, 1)$,which will be referred to by $''Kw''$ in the sequel.Its *cdf* is given by

$$F(x) = 1 - (1 - x^a)^b, x \in (0, 1), \tag{1.3}$$

where $a > 0$ and $b > 0$ are shape parameters. The model in (1.3) compares extremely favorably in terms of simplicity with the beta cdf, that is, the incomplete beta function ratio. The pdf corresponding to 1.3 is given by,

$$f(x) = ab(1 - x^a)^{b-1}, x \in (0, 1), \tag{1.4}$$

The Kw density function has the same basic shape properties of the beta distribution: $a > 1 and b > 1$ (unimodal); $a < 1 and b < 1$ (uniantimodal); $a > 1 and b \leqslant 1$ (increasing); $a \leqslant 1 and b > 1$(decreasing); $a = b = 1$ (constant). The Kw distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before. However, Jones (2009) explored the background and genesis of the Kw distribution and, more importantly, highlighted some advantages and disadvantages of the beta and Kw distributions.

For an arbitrary baseline *cdf*, $G(x)$, Cordeiro and Castro (2011) defined the Kw-G distribution by the *pdf* $f(x)$ and *cdf* $F(x) as$

$$f(x) = a\, b\, g(x)\, G^{a-1}(x)(1 - G^a(x))^{b-1}, \tag{1.5}$$

and

$$F(x) = 1 - (1 - G^a(x))^b, \tag{1.6}$$

respectively, where $g(x) = dG(x)/dx and a and b$ are two extra positive shape parameters. It follows immediately from (1.6) that the $Kw - G$distribution with parent *cdf* $G(x) = x$ produces the mini max distribution 1.3. If X is a random variable with pdf 1.5, we write $X\ Kw - G(a, b)$, where a and b are additional shape parameters which aim to govern skewness and tail weight of the generated distribution. An alluring element of this model is that the two parameters a and b can manage the cost of more prominent control over the weights in both tails and in its middle, Al-Babtain et al. (2015).

The rest of the article is organized as follows. In Section2, introduces the proposed Kumaraswamy new transmuted exponential model according to the new class of distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The Expansion for the pdf and the cdf Functions is derived in Section 4. In section 5, The statistical properties include quantile functions, median , moments , and moment generating function are given,. In Section 6, order statistics are discussed. In Section 7, we introduce the method of likelihood estimation as point estimation and, give the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. Finally, we fit the distribution to two real data sets to examine it and to suitability it with nested and non-nested models.

2 Kumaraswamy New Transmuted Exponential Distribution.

A modified transmuted exponential (MTE) distribution with cumulative distribution function (cdf) denoted by $G(x, \lambda, \beta, \alpha, \delta) \equiv G(x)$and probability density function $(pdf)(forx > 0)$ given by

$$G(x) = (1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}, \tag{2.1}$$

and the pdf

$$g(x) = \beta e^{-\beta x}][(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}],\ x > 0. \tag{2.2}$$

where $\alpha, \delta, \beta > 0\ if\ -1 < \lambda < 0\ and\ \alpha, \beta > 0, \frac{\alpha}{2} \leqslant \delta \leqslant \frac{5\alpha}{4}\ if\ 0 < \lambda < 1$. by inserting (2.1) into (1.6). Then the cumulative distribution function of $Kw - NTE$ model $(for\ x > 0)$ denoted by$F(x, \lambda, \beta, a, b, \alpha, \delta) \equiv F(x)$ becomes

$$F(x) = 1 - (1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b}, \tag{2.3}$$

whereas its *pdf* can be expressed, from (2.1),(2.2) and (1.5) as

$$\begin{aligned} f(x) = {} & ab\beta e^{-\beta x}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}] \\ & \times [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a-1} \\ & \times (1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b-1}. \end{aligned} \tag{2.4}$$

A physical interpretation of 2.3 is possible when a and b are positive integers. Suppose a system is made up of b independent components in series and that each component is made up of a independent sub-components in parallel. So, the system fails if any of the b components fail and each component fails if all of its a subcomponents fail. If the sub-component lifetimes have a common NTE cumulative function, then the lifetime of the entire system will follow the$Kw - NTE$distribution 2.3.

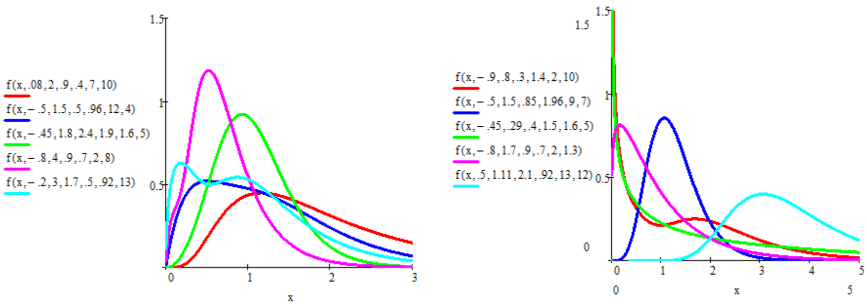
Furthermore, we can interpret the system from the redundancy view. Redundancy is a common method to increase reliability in an engineering design. Barlow and Proschan (1981) indicate that, if we want to increase the reliable of a given system, then redundancy at a component level is more effective than redundancy at a system level. That is, if all components of a system are available in duplicate, it is better to put these component pairs in parallel than it is to build two identical systems and place the systems in parallel.

The importance of the proposed $Kw - NTE$ model that it is flexible model that approaches to different distributions when its parameters are changed. The flexibility of the $Kw - NTE$ is explained in (Table 1) when their parameters are carefully chosen.

Table 1: The special cases of the $Kw-NTE$ distribution.

	Parameters						
Distribution	λ	β	a	b	α	δ	Author
$Kw-E$	0					1	
$Kw-EE$	0						
$Kw-TEE$						$\frac{\alpha}{2}$	
MTE			1	1			Mansour et al. (2015)
E	0		1	1		1	
TE			1	1	2	1	
EE	0		1	1	0		Gupta and Kundu (1999)
EE			1	1		$\frac{\alpha}{2}$	Merovci(2013)

Figures 1 and 2 illustrates some of the possible shapes of the *pdf* and *cdf* of the $Kw-NTE$ distribution for selected values of the parameters $\lambda, \beta, a, b, \delta\, and\, \alpha$ respectively.

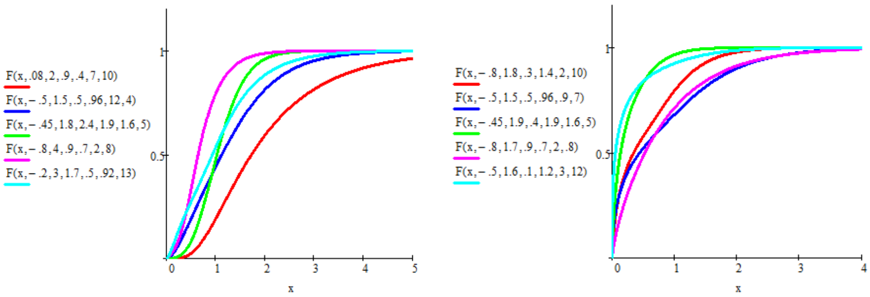


(a) (b)

Figure 1: Probability Density Function of the $Kw-NTE$ distribution.

3 Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF) and the cumulative hazard rate function (CHF) for the$Kw-NTE$are introduces in this section.



(a) (b)

Figure 2: Distribution Function of the $Kw - NTE$ distribution.

3.1 Reliability Function

The reliability function (RF) also known as the survival function, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. The reliability function of the Kumaraswamy new transmuted Exponential distribution $Kw{-}NTE$ denoted by $R_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta)$, can be a useful characterization of life time data analysis. It can be defined as,

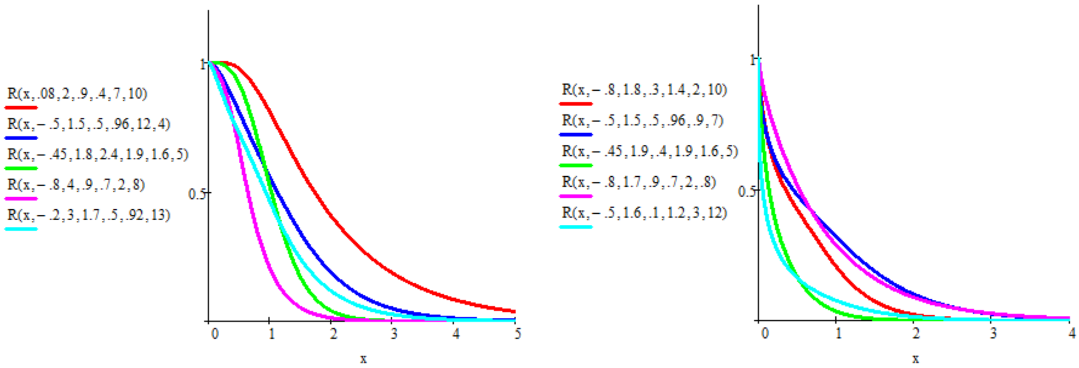
$$R_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta) = 1 - F_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta)$$

$$R_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta) = \left(1 - \left[(1+\lambda)\left[1 - e^{-\beta x}\right]^{\delta} - \lambda\left[1 - e^{-\beta x}\right]^{\alpha}\right]^{a}\right)^{b}. \tag{3.1}$$

Figure 3 illustrates the pattern of the called the Kumaraswamy new transmuted Exponential ($Kw - NTE$) distribution reliability function with different choices of parameters $\lambda, \beta, a, b, \delta$ *and* α

3.2 Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function (HF). the Kumaraswamy new transmuted Exponential distribution also known as instantaneous failure rate denoted by $h_{Kw-NTE}(x)$, is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The HF of the MTE is defined by

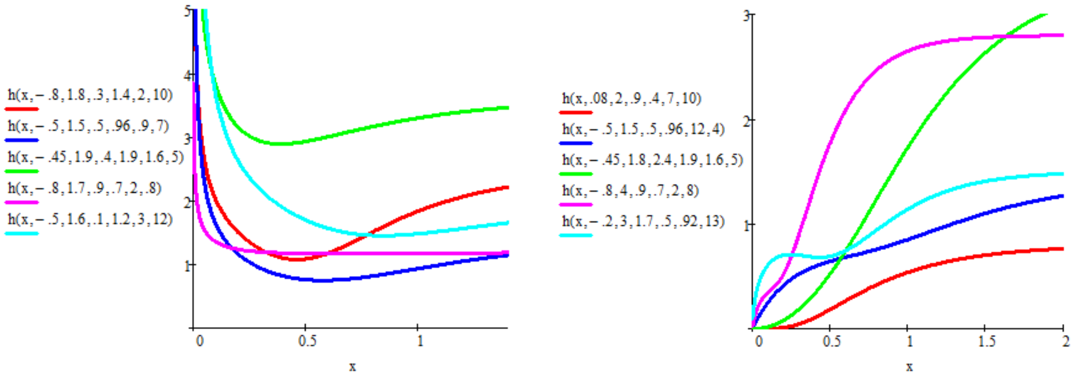


(a) (b)

Figure 3: Reliability Function of the $Kw - NTE$ distribution.

$$h_{Kw-NTE}(\lambda,\beta,a,b,\alpha,\delta) = ab\beta e^{-\beta x}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}] \times \frac{[(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a-1}}{1-[(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}}. \quad (3.2)$$

Figure 4 illustrates some of the possible shapes of the hazard rate function of the Kumaraswamy new transmuted Exponential distribution for different values of the parameters $\lambda, \beta, a, b, \delta\ and\ \alpha$.



(a) (b)

Figure 4: Hazard Rate of the $Kw-NTE$ distribution.

The $Kw-NTE$ model due to its flexibility in accommodating all forms of the hazard rate function as seen from Figure 4 (by changing its parameter values) seems to be an important distribution that can be used.

3.3 Cumulative Hazard Rate Function

The cumulative hazard function (CHF) of the modified transmuted exponential distribution, denoted by $H_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta)$ is defined as

$$H_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta) = \int_0^x h_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta)dx = -\ln R_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta),$$

$$H_{Kw-NTE}(\lambda, \beta, a, b, \alpha, \delta) = -\ln((1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b}). \tag{3.3}$$

4 Expansion for the *pdf* and the *cdf* Functions

In this section we introduced another expression for the pdf and the *cdf* functions using. The Maclaurin expansion to simplifying the *pdf* and the *cdf* forms.

4.1 Expansion for the pdf Function

From equation 2.4 and using the expansions

$$(1-z)^{k-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(k)}{\Gamma(k-1) j!} z^j. \tag{4.1}$$

Which holds for $|z| < 1$ and $k > 0$.

Using 4.1 and applying it to the term $\left(1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}\right)^{b-1}$, the pdf of the $Kw-NTE$ model can be written as

$$f(x) = a\, b\, \beta\, e^{-\beta x} \sum_{i=0}^{\infty} \frac{(-1)^{i}\Gamma(b)}{\Gamma(b-1)i!}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}]$$
$$\times (1+\lambda)^{a(i+1)-1}(1-e^{-\beta x})^{\delta(a(i+1)-1)}\left[1 - \frac{\lambda[1-e^{-\beta x}]^{\alpha}}{(1+\lambda)[1-e^{-\beta x}]^{\delta}}\right]^{a(i+1)-1}, \quad (4.2)$$

which holds for $\left|\frac{\lambda[1-e^{-\beta x}]^{\alpha}}{(1+\lambda)[1-e^{-\beta x}]^{\delta}}\right| < 0$.

Using 4.1 and applying it to the term $\left[1 - \frac{\lambda[1-e^{-\beta x}]^{\alpha}}{(1+\lambda)[1-e^{-\beta x}]^{\delta}}\right]^{a(i+1)-1}$ in 4.2,the pdf of the $Kw-NTE$ model can be written as

$$f(x) = a\, b\, \beta\, e^{-\beta x} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{(i+j)}\Gamma(b)\Gamma(a(i+1))}{i!j!\Gamma(b-1)\Gamma(a(i+1)-j)}$$
$$\times [(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}]$$
$$\times \lambda^{j}(1+\lambda)^{a(i+1)-j-1}(1-e^{-\beta x})^{\delta(a(i+1)-j-1)+\alpha j-\delta j}. \quad (4.3)$$

Using Binomial expansion and applying it to the term $[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}]$ in4.3,the pdf of the $Kw-NTE$ model can be written as

$$f(x) = a\, b\, \beta\, e^{-\beta x} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{1} \frac{(-1)^{(i+j+k)}\Gamma(b)\Gamma(a(i+1))}{i!j!\Gamma(b-1)\Gamma(a(i+1)-j)}$$
$$\times \alpha^{k}\delta^{1-k}\lambda^{j+k}(1+\lambda)^{a(i+1)+(1-k)-j-1}$$
$$\times (1-e^{-\beta x})^{\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k}. \quad (4.4)$$

Using 4.1 and applying it to the term$(1-e^{-\beta x})^{\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k}$ in 4.4,the pdf of the $Kw-NTE$ model can be written as

$$f(x) = a\, b\, \beta \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} \frac{(-1)^{(i+j+k)}\Gamma(b)\Gamma(a(i+1))}{i!j!\Gamma(b-1)\Gamma(a(i+1)-j)}$$
$$\times \frac{\Gamma(\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k+1)}{\Gamma(\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k-l+1)}$$
$$\times \alpha^{k}\delta^{1-k}\lambda^{j+k}(1+\lambda)^{a(i+1)+(1-k)-j-1}e^{-\beta x(l+1)}. \quad (4.5)$$

The *pdf* of $Kw-NTE$ distribution can then be represented as

$$f(x) = \sum_{i,j,l=0}^{\infty} \sum_{k=0}^{1} A_{i:k} e^{-\beta x(l+1)}, \tag{4.6}$$

where $A_{i:k}$ is a constant term given by

$$\begin{aligned} A_{i:k} =& a\, b\, \beta\, \frac{(-1)^{(i+j+k)} \Gamma(b)\Gamma(a(i+1))}{i!j!\Gamma(b-1)\Gamma(a(i+1)-j)} \\ & \times \frac{\Gamma(\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k+1)}{\Gamma(\delta(a(i+1)-j-1)+\alpha j-\delta j+(\delta-1)(1-k)+(\alpha-1)k-l+1)} \\ & \times \alpha^k \delta^{1-k} \lambda^{j+k} (1+\lambda)^{a(i+1)+(1-k)-j-1}. \end{aligned}$$

4.2 4.2 Expansion for the *cdf* Function

Using expansion 4.1 to Equation 4.2,then the *cdf* function of the $Kw-NTE$ can be written as:

$$F(x) = \sum_{i,j,k=0}^{\infty} B_{i:k} e^{-\beta k x}, \tag{4.7}$$

where $B_{i:k}$ is a constant term given by:

$$\begin{aligned} B_{i:k} =& \frac{(-1)^{(i+j+k)} \Gamma(b+1)\Gamma(ai+1)\Gamma(\delta(ai-j)+\alpha j+1)}{i!j!k!\Gamma(b-i+1)\Gamma(ai-j+1)\Gamma(delta(ai-j)+\alpha j-k+1} \\ & \times \lambda^j (1+\lambda)^{ai-j}. \end{aligned}$$

4.3 Statistical properties

In this section we discuss the most important statistical properties of the $Kw-NTE$ distribution.

4.4 Quantile function

The quantile function is obtained by inverting the cumulative distribution 4.7, where the $p-th$ quantile $x_p$of the $Kw-NTE$ model is the real solution of the following equation:

$$1 - \sum_{i,j,k=0}^{\infty} B_{i:k} e^{-\beta k x_p} - p = 0.$$

An expansion for the median M follows by taking $p = 0.5$.

4.5 5.2 Moments

The r^{th}non-central moments or (moments about the origin) of the $Kw-NTE$ under using equation 4.6 is given by

$$\mu\prime_r = E(x^r) = \int_0^\infty x^r f(x)dx,$$

$$\mu\prime_r = \int_0^\infty x^r \Big[\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} A_{i:k} e^{-\beta x(l+1)}\Big]dx,$$

then,

$$\mu\prime_r = \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} A_{i:k}\frac{\Gamma(r+1)}{\big(\beta(l+1)\big)^{r+1}}. \tag{4.8}$$

In particular, when $r=1$, Eq.4.8 yields the mean of the $Kw-NTE$ distribution, μ , as

$$\mu = \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} A_{i:k}\frac{1}{\big(\beta(l+1)\big)^{2}},$$

The n^{th}central moments or (moments about the mean) can be obtained easily from the r^{th} non-central moments throw the relation:

$$m_u = E(x-\mu)^n = \sum_{r=0}^{n}(-\mu)^{(n-r)}E(x^r).$$

Then the n^{th} central moments of the $Kw-NTE$ is given by,

$$m_u = \sum_{r=0}^{n}\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} A_{i:k}\frac{\Gamma(r+1)}{\big(\beta(l+1)\big)^{r+1}}(-\mu)^{(n-r)}.$$

4.6 The Moment Generating Function

The moment generating function,$M_x(t)$, can be easily obtained from the r^{th}non-central moment through the relation

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu\prime_r,$$

or

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx,$$

$$M_x(t) = \int_0^\infty e^{tx}\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1} A_{i:k} e^{-\beta x(l+1)}dx,$$

Then, the moment generating function of the $Kw-NTE$ distribution is given by,

$$M_x(t) = \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1}\frac{A_{i:k}}{\beta(l+1)+t}.$$

5 Order Statistics

Let $X_1, X_2, , X_n$denote n-independent random variables from a distribution function $F_X(x)$ with *pdf* $f_X(x)$. Let $X_{(1)}, X_{(2)}, , X_{(n)}$be the ordered sample arrangement. The pdf of $X_{(j)}$ is given by,

$$f_{X_j}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) \left[F_X(x)\right]^{(j-1)} \left[1 - F_X(x)\right]^{(n-j)}, j = 1, 2, ..., n.$$

Then from 2.3 and 2.4 the pdf of $X_(j)$ is given by:

$$\begin{aligned} f_{X_j}(x) =& \frac{n!}{(j-1)!(n-j)!} ab\beta e^{-\beta x}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}] \\ &\times [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a-1} \\ &\times (1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b-1} \\ &\times \left(1 - \left(1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}\right)^{b}\right)^{j-1} \\ &\times \left[1 - \left(1 - \left(1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}\right)^{b}\right)\right]^{n-j}. \end{aligned}$$

Therefore, the *pdfs* of the smallest and the largest order statistic are respectively given by:

$$\begin{aligned} f_{X_{(1)}}(x) =& nab\beta e^{-\beta x}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}] \\ &\times [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a-1} \\ &\times (1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b-1} \\ &\times \left[1 - \left(1 - \left(1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}\right)^{b}\right)\right]^{n-1}, \end{aligned}$$

and

$$\begin{aligned} f_{X_{(n)}}(x) =& nab\beta e^{-\beta x}[(1+\lambda)\delta[1-e^{-\beta x}]^{\delta-1} - \lambda\alpha[1-e^{-\beta x}]^{\alpha-1}] \\ &\times [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a-1} \\ &\times (1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a})^{b-1} \\ &\times \left(1 - \left(1 - [(1+\lambda)[1-e^{-\beta x}]^{\delta} - \lambda[1-e^{-\beta x}]^{\alpha}]^{a}\right)^{b}\right)^{n-1}. \end{aligned}$$

6 Estimation of the Parameters

In this section we introduce the method of likelihood to estimate the parameters involved then gives the equation used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques.

6.1 Maximum Likelihood Estimation

The maximum likelihood estimators ($MLEs$) for the parameters of Kumaraswamy new transmuted exponential distribution $Kw-NTE(\lambda,\beta,a,b,\alpha,\delta)$ is discussed in this section. Consider the random sample of size n from $Kw-NTE(\lambda,\beta,a,b,\alpha,\delta)$ with probability density function in 2.4, then the likelihood function can be expressed as follows

$$L(x_1,x_2,...,x_n,\lambda,\beta,a,b,\alpha,\delta)=L=\prod_{i=1}^{n} f_{Kw-NTE}(x_i,\lambda,\beta,a,b,\alpha,\delta),$$

$$\begin{aligned}L=&\prod_{i=1}^{n} ab\beta e^{-\beta x_i}[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}]\\ &\times\prod_{i=1}^{n}[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a-1}\\ &\times\prod_{i=1}^{n}(1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})^{b-1}.\end{aligned} \tag{6.1}$$

Hence, the log-likelihood function $\zeta=lnL$ becomes,

$$\begin{aligned}\zeta=&n\ln(a)+n\ln(b)+n\ln(\beta)-\sum_{i=1}^{n}\beta x_i\\ &+\sum_{i=1}^{n}\ln\Big[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}\Big]\\ &+\sum_{i=1}^{n}(a-1)\ln\Big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\Big)\\ &+\sum_{i=1}^{n}(b-1)\ln\Big((1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})\Big).\end{aligned} \tag{6.2}$$

Differentiating Equation 6.2 with respect to $\lambda,\beta,a,b,\alpha\ and\ \delta$ then equating it to zero, we obtain the $MLEs$ of $\lambda, \beta, a, b, \alpha \; and \; \delta$ as follows,

$$\begin{aligned}\frac{\partial \zeta}{\partial \lambda} =& \sum_{i=1}^{n}\Big[\frac{\delta[1-e^{-\beta x_i}]^{\delta-1}-\alpha[1-e^{-\beta x_i}]^{\alpha-1}}{[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}]}\Big]\\ &+\sum_{i=1}^{n}\Big[\frac{(a-1)\big[[1-e^{-\beta x_i}]^{\delta}-[1-e^{-\beta x_i}]^{\alpha}\big]}{\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)}\Big]\\ &-\sum_{i=1}^{n}\Big[\frac{a(b-1)\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)^{a-1}}{(1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})}\Big]\\ &\times\big([1-e^{-\beta x_i}]^{\delta}-[1-e^{-\beta x_i}]^{\alpha}\big),\end{aligned} \tag{6.3}$$

$$\begin{aligned}\frac{\partial \zeta}{\partial \beta} =& \frac{n}{\beta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\Big[\frac{x_i e^{-\beta x_i}\big[(1+\lambda)\delta(\delta-1)[1-e^{-\beta x_i}]^{\delta-2}-\lambda\alpha(\alpha-1)[1-e^{-\beta x_i}]^{\alpha-2}\big]}{\big[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}\big]}\Big]\\ &+\sum_{i=1}^{n}\Big[\frac{(a-1)x_i e^{-\beta x_i}\big[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}\big]}{\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)}\Big]\\ &-\sum_{i=1}^{n}\Big[\frac{(b-1)a x_i e^{-\beta x_i}[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a-1}}{(1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})}\Big],\end{aligned} \tag{6.4}$$

$$\begin{aligned}\frac{\partial \zeta}{\partial a} =& \frac{n}{a}+\sum_{i=1}^{n}\ln\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)\\ &-\sum_{i=1}^{n}\Big[\frac{(b-1)[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})}{(1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a})}\Big]\\ &\times\ln\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big),\end{aligned} \tag{6.5}$$

$$\frac{\partial \zeta}{\partial b} = \frac{n}{a}+\sum_{i=1}^{n}\ln\big[1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a}\big], \tag{6.6}$$

$$\begin{aligned}\frac{\partial \zeta}{\partial \alpha} =& \sum_{i=1}^{n}\Big[\frac{(-\lambda)\alpha[1-e^{-\beta x_i}]^{\alpha-1}[\alpha\ln([1-e^{-\beta x_i}])+1]}{[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}]}\Big]\\ &-\sum_{i=1}^{n}(a-1)\Big[\frac{(\lambda)[1-e^{-\beta x_i}]^{\alpha}\ln[1-e^{-\beta x_i}]}{\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)}\Big]\\ &+\sum_{i=1}^{n}\Big[\frac{(b-1)a(\lambda)[1-e^{-\beta x_i}]^{\alpha}\ln[1-e^{-\beta x_i}]}{[1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a}]}\Big]\\ &\times[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a-1},\end{aligned} \tag{6.7}$$

and

$$\frac{\partial\zeta}{\partial\delta} = \sum_{i=1}^{n}\Bigg[\frac{(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}[\delta\ln([1-e^{-\beta x_i}])+1]}{[(1+\lambda)\delta[1-e^{-\beta x_i}]^{\delta-1}-\lambda\alpha[1-e^{-\beta x_i}]^{\alpha-1}]}\Bigg]$$
$$+\sum_{i=1}^{n}(a-1)\Bigg[\frac{(1+\lambda)[1-e^{-\beta x_i}]^{\delta}\ln[1-e^{-\beta x_i}]}{\big((1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}\big)}\Bigg]$$
$$-\sum_{i=1}^{n}\Bigg[\frac{(b-1)a(1+\lambda)[1-e^{-\beta x_i}]^{\delta}\ln[1-e^{-\beta x_i}]}{[1-[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a}]}\Bigg]$$
$$\times\,[(1+\lambda)[1-e^{-\beta x_i}]^{\delta}-\lambda[1-e^{-\beta x_i}]^{\alpha}]^{a-1}. \tag{6.8}$$

The maximum likelihood estimator $\underline{\hat{\vartheta}} = (\hat{\lambda}, \hat{\beta}, \hat{a}, \hat{b}, \hat{\alpha}, \hat{\delta})$ of $\underline{\vartheta} = (\lambda, \beta, a, b, \alpha, \delta)$ is obtained by solving the nonlinear system of equations 6.3 through 6.7. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

6.2 Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin (1983). This method is based on an idea that the differences (Spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{6.9}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\lambda,\beta,a,b,\alpha,\delta)dx \ : \quad i = 1,2,...,n+1, \tag{6.10}$$

where, $F(x_{(0)},\lambda,\beta,a,b,\alpha,\delta) = 0$ and $F(x_{(n+1)},\lambda,\beta,a,b,\alpha,\delta) = 0$. The MPS estimators $\hat{\lambda}_{ps}, \hat{\beta}_{ps}, \hat{a}_{ps}, \hat{b}_{ps}, \hat{\delta}_{ps}\ and\ \hat{\alpha}_{ps}$ of $\lambda,\beta,a,b,\delta\ and\ \alpha$ are obtained by maximizing the geometric mean (GM) of the differences. Substituting *pdf* of $Kw-NTE$ distribution in 6.10 and taking logarithm of the above expression, we will have

$$\log GM = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\Bigg[F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta) - F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)\Bigg]. \tag{6.11}$$

The MPS estimators $\hat{\lambda}_{ps}, \hat{\beta}_{ps}, \hat{a}_{ps}, \hat{b}_{ps}, \hat{\delta}_{ps}\ and\ \hat{\alpha}_{ps}$ of $\lambda,\beta,a,b,\delta\ and\ \alpha$ can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial logGM}{\partial\lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\Bigg[\frac{F\prime_{\lambda}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta) - F\prime_{\lambda}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta) - F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\Bigg] = 0,$$

$$\frac{\partial logGM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F\prime_{\beta}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F\prime_{\beta}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\right]=0,$$

$$\frac{\partial logGM}{\partial a} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F\prime_{a}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F\prime_{a}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\right]=0,$$

$$\frac{\partial logGM}{\partial b} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F\prime_{b}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F\prime_{b}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\right]=0,$$

$$\frac{\partial logGM}{\partial \delta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F\prime_{\delta}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F\prime_{\delta}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\right]=0,$$

and

$$\frac{\partial logGM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F\prime_{\alpha}(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F\prime_{\alpha}(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}{F(x_{(i)},\lambda,\beta,a,b,\alpha,\delta)-F(x_{(i-1)},\lambda,\beta,a,b,\alpha,\delta)}\right]=0.$$

6.3 Least square estimates

Let $x_{(1)}, x_{(2)}, , x_{(n)}$ be the ordered sample of size n drawn the $Kw-NTE$ distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F(X_{(i)})\right] = \frac{i}{n+1} \quad i=1,2,...,n. \tag{6.12}$$

The least square estimates $\hat{\lambda}_{LS}, \hat{\beta}_{LS}, \hat{a}_{LS}, \hat{b}_{LS}, \hat{\delta}_{LS} \quad and \quad \hat{\alpha}_{LS}$ of $\lambda, \beta, a, b, \delta \quad and \quad \alpha$ are obtained by minimizing

$$Z(\lambda,\beta,a,b,\delta,\alpha) = \sum_{i=1}^{n+1}\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)-\frac{i}{n+1}\right]^2.$$

Therefore, $\hat{\lambda}_{LS}, \hat{\beta}_{LS}, \hat{a}_{LS}, \hat{b}_{LS}, \hat{\delta}_{LS} and \hat{\alpha}_{LS}$ of $\lambda, \beta, a, b, \delta and \alpha$ can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\lambda,\beta,a,b,\delta,\alpha)}{\partial \lambda} = \sum_{i=1}^{n+1} F\prime_{\lambda}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)-\frac{i}{n+1}\right]=0,$$

$$\frac{\partial Z(\lambda,\beta,a,b,\delta,\alpha)}{\partial \beta} = \sum_{i=1}^{n+1} F\prime_{\beta}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)-\frac{i}{n+1}\right]=0,$$

$$\frac{\partial Z(\lambda,\beta,a,b,\delta,\alpha)}{\partial a} = \sum_{i=1}^{n+1} F\prime_{a}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)-\frac{i}{n+1}\right]=0,$$

$$\frac{\partial Z(\lambda,\beta,a,b,\alpha,\alpha)}{\partial b} = \sum_{i=1}^{n+1} F\prime_{b}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)-\frac{i}{n+1}\right]=0,$$

$$\frac{\partial Z(\lambda,\beta,a,b,\delta,\alpha)}{\partial\delta} = \sum_{i=1}^{n+1} F\prime_{\delta}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha) - \frac{i}{n+1}\right] = 0,$$

and

$$\frac{\partial Z(\lambda,\beta,a,b,\alpha,\alpha)}{\partial\alpha} = \sum_{i=1}^{n+1} F\prime_{\alpha}(x_{(i)},\lambda,\beta,a,b,\delta,\alpha)\left[F(x_{(i)},\lambda,\beta,a,b,\delta,\alpha) - \frac{i}{n+1}\right] = 0.$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R. We used nlm() package for optimizing 6.2.

7 Simulation algorithms

In this section we give an algorithm, using R software, to simulate data from the $Kw-NTE$ model.

7.1 Inverse CDF method

Since the probability integral transformation cannot be applied explicitly, we, therefore need to follow the following steps for generating a sample of size n from $Kw-NTE$ $(\lambda, \beta, a, b, \alpha, \alpha)$:
Step1: Set $n, \lambda, \beta, a, b, \alpha, \alpha$ and initial value x^0.
Step2: Generate$U \sim$ Uniform(0,1).
Step3: Update x^0 by using the Newton's formula

$$x^* = x^0 - R(x^0, \Theta),$$

where, $R(x^0, \Theta) = \frac{F_x(x^0,\Theta)-U}{f_x(x^0,\Theta)}$, $F_x(.)\, and\, f_x(.)$ are *cdf* and pdf of $Kw-NTE$ distribution, respectively.
Step4: If $|x^0 - x^*| \leq \varepsilon$, (very small, $\varepsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from $Kw-NTE$ distribution.
Step5: If $|x^0 - x^* > \varepsilon$,, then, set$x^0 = x^*$and go to step 3.
Step6: Repeat steps 3-5, n times for $x_1, x_2, , x_n$respectively.

7.2 Simulation Study

This subsection explores the behaviors of the proposed estimators in terms of their mean square error on the basis of simulated samples from *pdf* of $Kw-NTE$ with varying sample sizes. We take $\lambda = -0.55, \beta = 3, a = 2, b = 4, \delta = 3,\, and\, \alpha = 2$ arbitrarily and n=10(10)100. The algorithms are coded in R, and the algorithm given in 8.1 has been used for simulation purposes. We calculate MLE estimators of $\lambda, \beta, a, b, \delta$ and α based on each generated sample. This simulation is repeated 1000 of times, and average estimates with corresponding mean square errors are computed and reported in Table 2.

Table 2: Estimates and mean square errors (in 2-nd row of each cell) of the proposed estimators with varying sample size.

	MLE					
n	λ	β	a	b	δ	α
10	-0.5024	3.6501	2.0652	4.8801	3.0175	2.5544
	0.1270	1.9205	0.0885	1.9961	0.1300	1.2817
20	-0.5058	3.4466	2.0622	4.8455	3.0022	2.2391
	0.0489	1.4783	0.0722	0.4558	0.0507	0.3825
30	0.5406	2.9786	2.0381	4.8336	3.9949	2.1573
	0.0299	0.4971	0.0364	0.2553	0.0307	0.2298
40	-0.5634	2.8352	2.0146	4.8654	3.9952	2.1215
	0.0253	0.3510	0.0312	0.2419	0.0224	0.1590
50	-0.6013	2.7542	2.0203	4.8742	3.9954	2.0965
	0.0181	0.2685	0.0269	0.1802	0.0184	0.1252
60	-0.6101	2.8154	2.0106	4.9612	3.9956	2.0804
	0.0168	0.2379	0.0187	0.1379	0.0148	0.0998
70	-0.6310	2.7223	3.0101	4.9700	3.9966	2.0711
	0.0119	0.2175	0.0112	0.1175	3.0125	0.0872
80	-0.6627	2.7700	2.0100	4.9932	3.9978	2.0553
	0.0100	0.0109	0.0132	0.1089	4.0106	0.0671
90	-0.6688	2.6920	2.0104	4.9943	3.9992	2.0511
	0.0089	0.0089	0.0073	0.0784	3.0095	0.0619
100	-0.6802	2.6518	2.0087	4.9938	3.9982	2.0471
	0.0061	0.0085	0.0082	0.0675	0.0087	0.0545

From Table 2, it can be clearly observed that as sample size increases the mean square error decreases, which proves the consistency of the estimators.

8 Applications

In this section, we use two real data sets to compare the fits of the$Kw-NTE$distribution with three sub-models and Weibull model. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the R code.

8.1 Data Set-1

The first data set represents failure time of 50 items reported in Aarset (1987). In order to compare the two distribution models, we consider criteria like KS(Kolmogorov Smirnov), $-2L, AIC$ (Akaike information criterion), AIC_C(corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller KS, $-2L, AIC\ and\ AIC_C$ values:

$$AIC = -2L + 2k,$$

$$AIC_C = -2L + \left(\frac{2kn}{n-k-1}\right),$$

and

$$BIC = -2L + k\log(n),$$

where L denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size. Also, for calculating the values of KS we use the sample estimates of $\lambda, \beta, a, b, \delta$ andα . Table 3 shows the parameter estimation based on the maximum likelihood and gives the values of the criteria AIC, AIC_C, BIC, andKS test. The values in Table 3 indicate that the $Kw-NTE$ distribution leads to a better fit over all the other models.

Table 3: MLEs the measures $AIC, AIC_C, BIC,\ and\ KS$ test to failure time data for the models

Model	Parameter Estimates	Standard Error	$-LogL$	AIC	AIC_C	BIC	KS
Kw	$\hat{\lambda} = -0.957$	0.02620	217.6843	447.3686	448.4848	458.8407	0.1082
$-NTE$	$\hat{\beta} = 0.035$	0.0034					
	$\hat{a} = 107.4$	61.41					
	$\hat{b} = 75.964$	32.61					
	$\hat{\alpha} = 25.7$	0.003					
	$\hat{\delta} = 0.006$	15.25					
TEE	$\hat{\lambda} = -0.481$	0.2728	238.689	483.3793	483.9011	489.1154	0.1662
	$\hat{\beta} = 0.6594$	0.0038					
	$\hat{\alpha} = 0.0209$	0.0038					
EE	$\hat{\beta} = 2.61$	0.2381	239.9733	483.9467	484.2021	487.7708	0.1843
	$\hat{\alpha} = 31.34$	9.525					
W	$\hat{\beta} = 5.78$	0.5762	240.979	485.95	486.2145	489.7832	0.1729
	$\hat{\lambda} = 0.614$	0.0139					
E	$\hat{\beta} = 0.021$	0.0031	241.067	484.1354	484.218	486.0474	0.171

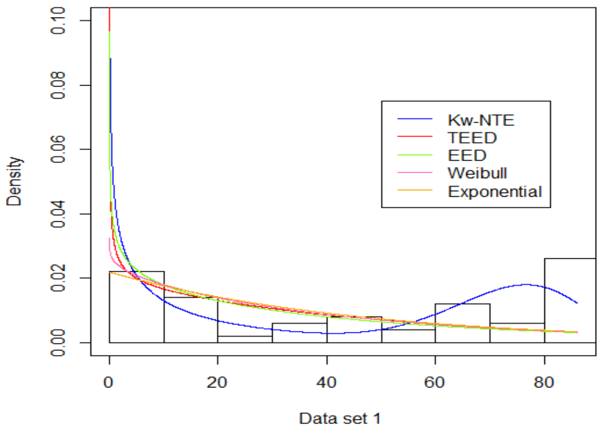


Figure 5: Estimated densities of data set 1.

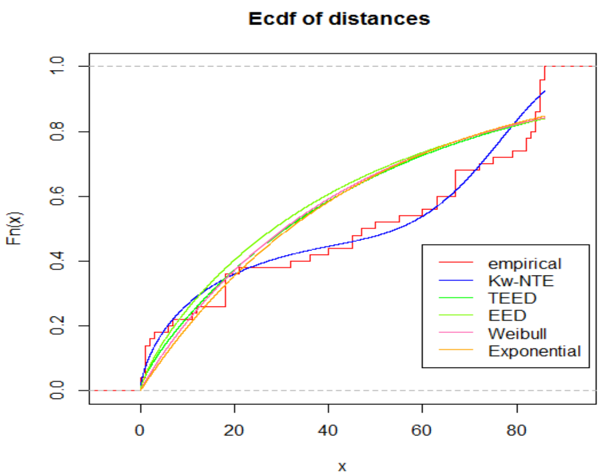


Figure 6: Empirical, fitted $Kw - NTE$, transmuted exponentiated exponential, exponentiated exponential , Weibull , and exponential distributions of data set 1.

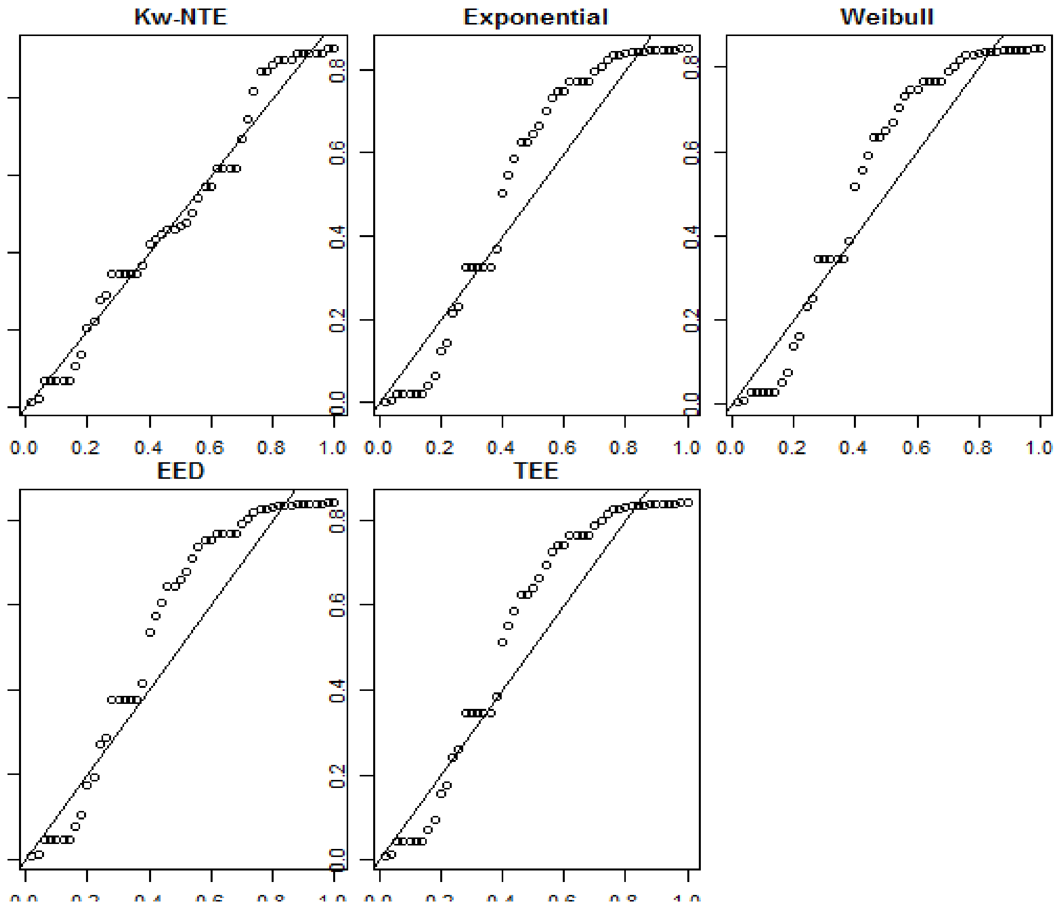


Figure 7: Probability plots for the fits $Kw - NTE$, transmuted exponentiated exponential, exponentiated exponential , Weibull , and exponential distributions 1.

8.2 Data Set 2

The second data set represents the ages for 155 patients of breast tumors taken from (June-November 2014), whose entered in (Breast Tumors Early Detection Unit, Benha Hospital University, Egypt)reported in Mansour et al (2015). The values in Table 4 indicate that the $Kw - NTE$ distribution leads to a better fit over all the other models.

Table 4: MLEs the measures $AIC, AIC_C, BIC, and KS$ test to 155 patients of breast tumors data for the models.

Model	Parameter Estimates	Standard Error	$-LogL$	AIC	AIC_C	BIC	KS
Kw	$\hat{\lambda} = -0.756$	0.116	598.35	1208.7	1209.03	1226.97	0.074
$-NTE$	$\hat{\beta} = 0.116$	0.019					
	$\hat{a} = 10.14$	5.522					
	$\hat{b} = 1.114$	0.363					
	$\hat{\alpha} = 45.83$	26.98					
	$\hat{\delta} = 1.115$	0.829					
TEE	$\hat{\lambda} = -0.77$	0.122	606.38	1218.7	1218.9	1227.90	0.10
	$\hat{\beta} = 21.88$	5.263					
	$\hat{\alpha} = 0.095$	0.0058					
EE	$\hat{\beta} = 0.086$	0.005	611.2	1226.4	1226.5	1232.5	0.11
	$\hat{\alpha} = 25.59$	4.974					
W	$\hat{\beta} = 3.687$	0.208	610.29	1224.5	1224.6	1230.68	0.13
	$\hat{\lambda} = 0.020$	0.0004					
E	$\hat{\beta} = 0.022$	0.0018	740.31	1482.6	1482.6	1485.6	0.408

These results indicate that the $Kw{-}NTE$ model has the lowest AIC, AIC_C, KSandBIC values among the fitted models. The values of these statistics indicate that the $Kw - NTE$ model provides the best fit to all of the data.

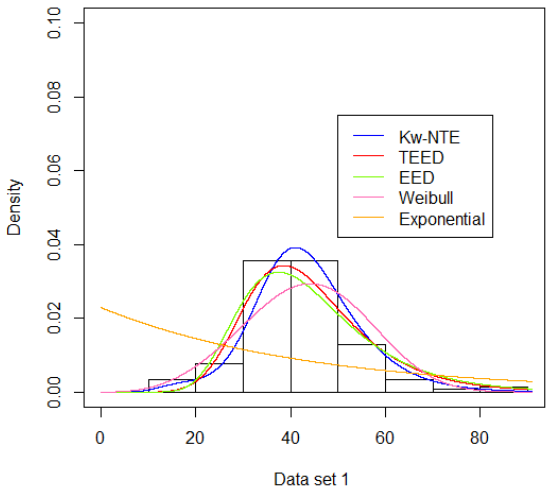


Figure 8: Estimated densities of data set 2.

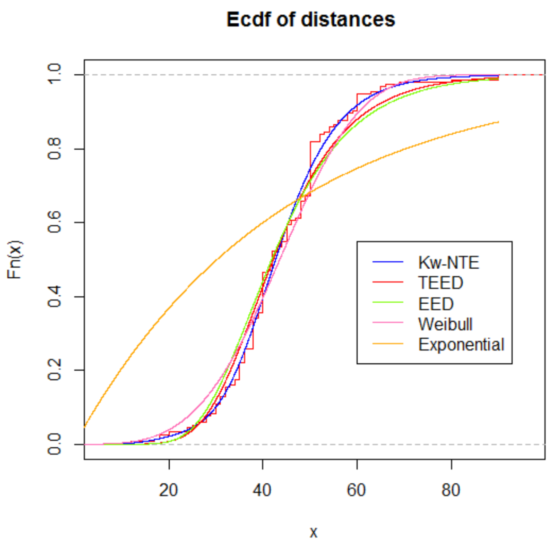


Figure 9: Empirical, fitted $Kw - NTE$, transmuted exponentiated exponential, exponentiated exponential , Weibull , and exponential distributions of data set 2.

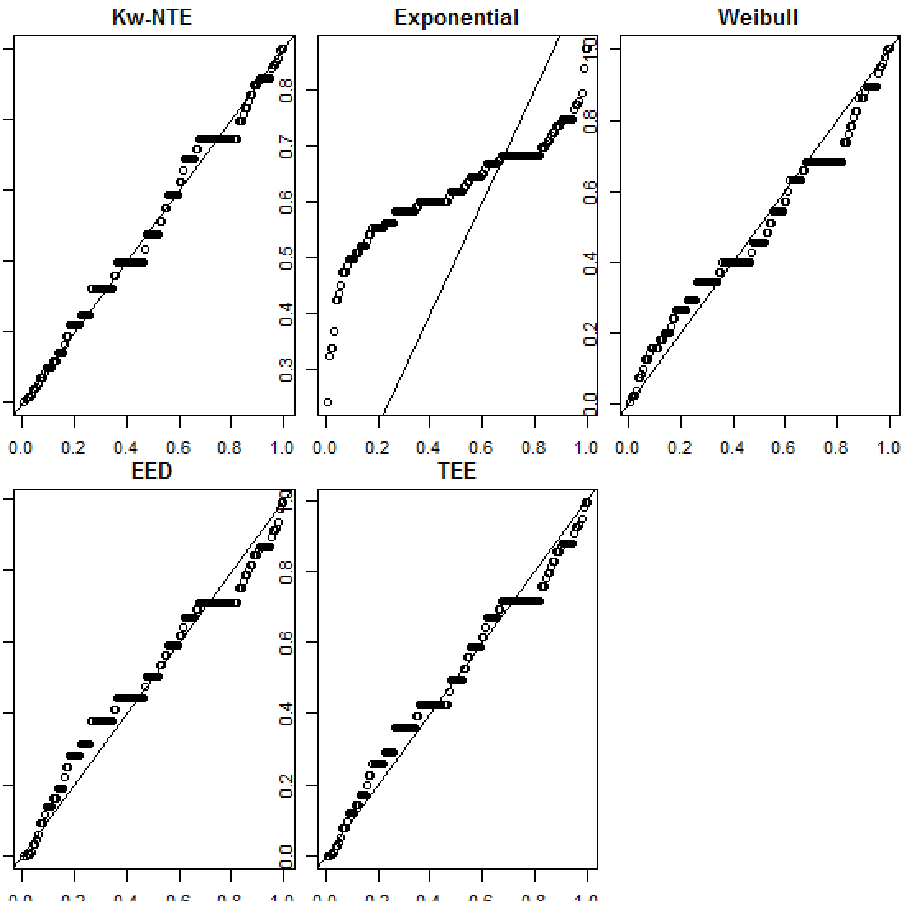


Figure 10: Probability plots for the fits of the $Kw - NTE$, transmuted exponentiated exponential, exponentiated exponential, Weibull, and exponential distributions of data set 2.

Concluding remarks

There has been an extraordinary enthusiasm among statisticians and connected specialists in developing adaptable lifetime models to encourage better demonstrating of survival information. Hence, a huge advancement has been made towards the speculation of some surely understood lifetime models and their fruitful application to issues in a few ranges. In this paper, we present another six-parameter model got utilizing the Kumaraswamy generalization technique. We refer to the new model as the $Kw - NTE$ distribution and study some of its mathematical and statistical properties. We provide the pdf, the *cdf* and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood and method of moment. The new model is compared with nested and non nested models and provides consistently better fit than other lifetime models. We hope that the proposed distribution will serve as an alternative model to other models available in the literature.

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